

# When does a biased graph come from a group labelling?

Matt DeVos\*

Daryl Funk†

Irene Pivotto‡

## Abstract

A biased graph consists of a graph  $G$  together with a collection of distinguished cycles of  $G$ , called balanced cycles, with the property that no theta subgraph contains exactly two balanced cycles. Perhaps the most natural biased graphs on  $G$  arise from orienting  $G$  and then labelling the edges of  $G$  with elements of a group  $\Gamma$ . In this case, we may define a biased graph by declaring a cycle to be balanced if the product of the labels on its edges is the identity, with the convention that we take the inverse value for an edge traversed backwards. Our first result gives a natural topological characterisation of biased graphs arising from group-labellings.

In the second part of this article, we use this theorem to construct some exceptional biased graphs. Notably, we prove that for every  $m \geq 3$  and  $\ell$  there exists a minor minimal not group labellable biased graph on  $m$  vertices where every pair of vertices is joined by at least  $\ell$  edges. Finally, we show that these results extend to give infinite families of excluded minors for certain families of frame and lift matroids.

## 1 Introduction

Throughout we shall assume that all graphs are finite, but may have loops and parallel edges. A *theta* graph consists of two distinct vertices  $x, y$  and three internally disjoint paths from  $x$  to  $y$ . Thus a theta graph has exactly three cycles. A *biased graph* consists of a pair  $(G, \mathcal{B})$  where  $G$  is a graph and  $\mathcal{B}$  is a collection of cycles, called *balanced*, obeying the *theta property*

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\*Simon Fraser University, mdevos@sfu.ca. Supported in part by an NSERC Discovery Grant (Canada).

†Simon Fraser University, dfunk@sfu.ca. Supported in part by an NSERC Postgraduate Scholarship.

‡University of Western Australia, irene.pivotto@uwa.edu.au. Supported by an Australian Research Council Discovery Project (project number DP110101596).

- that is, there does not exist a theta subgraph of  $G$  for which exactly two of the three cycles are balanced. Cycles not in  $\mathcal{B}$  are called *unbalanced*. We view ordinary graphs as a special case of biased graphs where every cycle is balanced.

The general theory of biased graphs was developed in a sequence of papers by Zaslavsky [5, 6, 7, 8, 9, 10, 11]. More recently, biased graphs have risen to prominence thanks to the central role they play in the matroid minors project (see, for example [1, 2, 4]).

Perhaps the most natural families of biased graphs arise from group labelled graphs (also called *gain* graphs). A *group labelling* of a graph  $G$  consists of an orientation of the edges of the graph together with a function  $\phi : E(G) \rightarrow \Gamma$ , where  $\Gamma$  is a group (written multiplicatively). Consider a walk  $W$  in the underlying graph of  $G$  with edge sequence  $e_1, e_2, \dots, e_\ell$  and define  $\epsilon_i$  using the orientation of  $G$  as follows

$$\epsilon_i = \begin{cases} 1 & \text{if } e_i \text{ is traversed forward in } W \\ -1 & \text{if } e_i \text{ is traversed backward in } W \end{cases}$$

Now we extend  $\phi$  by defining

$$\phi(W) = \prod_{i=1}^{\ell} \phi(e_i)^{\epsilon_i}.$$

Note that the *reverse* of  $W$ , which we denote  $W^{-1}$ , satisfies  $\phi(W^{-1}) = \phi(W)^{-1}$ . Furthermore, if  $W$  is a closed walk, and  $W'$  is another closed walk with the same cyclic sequence of edges (i.e.  $W'$  has sequence  $e_j, e_{j+1}, \dots, e_\ell, e_1, \dots, e_{j-1}$ ), then  $\phi(W)$  and  $\phi(W')$  will be conjugate. It follows from this that whenever a closed walk  $W$  has  $\phi(W) = 1$ , the same will be true for every closed walk with the same edge sequence and for the reverse of every such walk.

For a group labelling of  $G$  with function  $\phi$  we define  $\mathcal{B}_\phi$  to be the set of all cycles  $C$  of  $G$  for which some (and thus every) simple closed walk  $W$  around  $C$  satisfies  $\phi(W) = 1$ . We claim that  $(G, \mathcal{B}_\phi)$  is a biased graph. To verify this, consider a theta subgraph  $H \subseteq G$  consisting of three edge disjoint paths  $P_1, P_2, P_3$  from  $x$  to  $y$ . If the cycle  $P_1 \cup P_2$  is balanced, then  $\phi(P_1)\phi(P_2^{-1}) = 1$  so  $\phi(P_1) = \phi(P_2)$ . Similarly, if  $P_2 \cup P_3$  is balanced then  $\phi(P_2) = \phi(P_3)$ . These assumptions then imply  $\phi(P_1) = \phi(P_3)$  so the third cycle  $P_1 \cup P_3$  is also balanced.

We say that a biased graph  $(G, \mathcal{B})$  is  $\Gamma$ -*labellable* if there is a group labelling of  $G$  given by  $\phi : E(G) \rightarrow \Gamma$  so that  $(G, \mathcal{B}_\phi) = (G, \mathcal{B})$ . If  $(G, \mathcal{B})$  is  $\Gamma$ -labellable for some group  $\Gamma$  then we say it is *group labellable*. Our first result gives a topological criteria to determine if a biased graph is group labellable.

**Theorem 1.1.** *Let  $(G, \mathcal{B})$  be a biased graph and construct a 2-cell complex  $K$  from  $G$  by adding a disc with boundary  $C$  for every  $C \in \mathcal{B}$ . Then the following are equivalent.*

1.  $(G, \mathcal{B})$  is group labellable.
2. Every cycle  $C \notin \mathcal{B}$  is a non-contractible curve in  $K$ .

There is a natural notion of minor for biased graphs which extends the usual notion for graphs. For an edge  $e \in E(G)$  we *delete*  $e$  from  $(G, \mathcal{B})$  by deleting  $e$  from  $G$  and then removing from  $\mathcal{B}$  every cycle containing  $e$ . For a balanced loop  $e$ , the contraction  $(G, \mathcal{B})/e$  is defined as  $(G, \mathcal{B}) \setminus e$ . For a non-loop edge  $e$ , we *contract*  $e$  from  $(G, \mathcal{B})$  by contracting  $e$  in the graph and then declaring a cycle  $C$  to be *balanced* if either  $C \in \mathcal{B}$  or  $E(C) \cup \{e\}$  is the edge set of a cycle in  $\mathcal{B}$ . Let us pause to note that when working with frame and lift matroids based on a biased graph, we will permit contraction of unbalanced loop edges (and these matroids differ here). For now, however, while working with just biased graphs, we will disallow this operation. It is straightforward to verify that both deletion and contraction preserve the theta property, so these operations always yield a new biased graph. As usual, we define a *minor* of  $(G, \mathcal{B})$  to be any biased graph formed by a sequence of deletions and contractions.

If  $G$  is labelled with the group  $\Gamma$  by the function  $\phi : E(G) \rightarrow \Gamma$ , then there are a number of basic transformations we can apply to the orientation and group labelling which will have no effect on  $(G, \mathcal{B}_\phi)$ . For instance, we may reverse the orientation of the edge  $e$  and replace  $\phi(e)$  by its inverse. We call this operation an *edge reversal*. Similarly, if  $v \in V(G)$  and all non-loop edges incident with  $v$  are directed outward, then we may modify  $\phi$  on every such edge  $e$  by replacing  $\phi(e)$  by  $\phi(e)g$  for some fixed  $g \in \Gamma$ . We call this operation a *switch* at  $v$ . Any group labelled graph obtained by a sequence of reversals and switches is called *equivalent* for the natural reason that any two such labellings have the same set of balanced cycles. Using these operations easily gives the following property.

**Proposition 1.2.** *If a biased graph is  $\Gamma$ -labellable, then so is every minor.*

*Proof.* Choose an orientation and group labelling  $\phi : E(G) \rightarrow \Gamma$  so that  $\mathcal{B}_\phi = \mathcal{B}$ . To prove the result, it suffices to check that it holds under both edge deletion and edge contraction. To delete the edge  $e$ , we simply remove it from  $G$  and from the domain of  $\phi$ . To contract a non-loop edge  $e$  we first move via reversals and switches to an equivalent labelling  $\phi' : E(G) \rightarrow \Gamma$

for which  $\phi'(e) = 1$ . Then contract  $e$  in the graph  $G$  and remove it from the domain of  $\phi'$ .  $\square$

Using Theorem 1.1 we give a general construction for biased graphs which are minor minimal subject to not being group-labellable. Our next result is a consequence of this.

**Theorem 1.3.** *For every  $t \geq 3$  and  $\ell$  there exists a biased graph  $(G, \mathcal{B})$  with the following properties:*

1.  $G$  is a graph on  $t$  vertices and every pair of vertices is joined by at least  $\ell$  edges.
2.  $(G, \mathcal{B})$  is not group-labellable.
3. For every infinite group  $\Gamma$ , every proper minor of  $(G, \mathcal{B})$  is  $\Gamma$ -labellable.

For every group  $\Gamma$ , let  $\mathcal{G}_\Gamma$  denote the family of all biased graphs which can be  $\Gamma$ -labelled. Every  $\mathcal{G}_\Gamma$  is a minor closed class of biased graphs, so it is natural to ask about its set of excluded minors (*i.e.* the minor minimal biased graphs which are not  $\Gamma$ -labellable). Theorem 1.3 shows that for every infinite group  $\Gamma$  the class  $\mathcal{G}_\Gamma$  has a rich set of excluded minors. In particular, we have the following obvious consequence.

**Corollary 1.4.** *For every infinite group  $\Gamma$  and every  $t \geq 3$  there are infinitely many excluded minors for  $\mathcal{G}_\Gamma$  with exactly  $t$  vertices.*

For both graphs and biased graphs, there are natural partial orders defined by the rule that a graph  $G$  (biased graph  $(G, \mathcal{B})$ ) dominates another graph  $H$  (biased graph  $(H, \mathcal{C})$ ) if and only if  $H$  ( $(H, \mathcal{C})$ ) is isomorphic to a minor of  $G$  ( $(G, \mathcal{B})$ ). A famous theorem due to Robertson and Seymour asserts that for graphs, this partial order has no infinite antichain (equivalently, every proper minor closed class of graphs is characterized by a finite list of excluded minors). In contrast, the above result shows that the partial order for biased graphs has infinite antichains with each member on a fixed number of vertices.

In fact, there are some very simple infinite antichains of biased graphs. For instance, let  $2C_n$  denote the graph obtained from a directed cycle of length  $n$  by adding an edge in parallel with every existing edge. Let  $\mathcal{B}_n$  consist of two edge disjoint cycles of length  $n$  in the graph  $2C_n$ . Then each  $(2C_n, \mathcal{B}_n)$  is a biased graph, and the set  $\{(2C_n, \mathcal{B}_n) \mid n \geq 3\}$  is an infinite antichain. To see this, note that each of these biased graphs has exactly two balanced cycles,

but contracting or deleting an edge gives a biased graph with fewer than two balanced cycles, and this will remain true under further deletions and contractions. In the last section we will show that for every infinite group  $\Gamma$ , all of these biased graphs are contained in  $\mathcal{G}_\Gamma$ . Further, we will prove the following result, showing that  $\mathcal{G}_\Gamma$  may also contain infinite antichains on a fixed number of vertices.

**Theorem 1.5.** *Let  $\Gamma$  be a group and let  $t \geq 3$ . There exists an infinite antichain of biased graphs in  $\mathcal{G}_\Gamma$  all on  $t$  vertices if and only if  $\Gamma$  is infinite.*

For each biased graph  $(G, \mathcal{B})$ , there are two matroids naturally associated with  $(G, \mathcal{B})$ , on ground set  $E(G)$ , the *lift* matroid  $L(G, \mathcal{B})$  and *frame* matroid  $F(G, \mathcal{B})$ . These were defined by Zaslavsky in [9]. These can be defined in terms of circuits as follows. A set  $C \subseteq E(G)$  is a circuit of the lift matroid  $L(G, \mathcal{B})$  if  $C$  is balanced, the union of two unbalanced cycles meeting in at most one vertex, or a theta subgraph containing no balanced cycle. A set  $C \subseteq E(G)$  is a circuit of the frame matroid  $F(G, \mathcal{B})$  if  $C$  is balanced, the union of two unbalanced cycles meeting in at most one vertex together with a path connecting them if these cycles are disjoint, or a theta subgraph containing no balanced cycle.

Spikes and swirls are two families of matroids that have been an important source of examples in studies of representability of matroids over fields. For each integer  $n \geq 3$ , a rank  $n$  *spike* is obtained by taking  $n$  concurrent three-point lines  $\{x_i, y_i, z\}$  ( $i \in \{1, \dots, n\}$ ) freely in  $n$ -space, then deleting their common point of intersection  $z$ . If no choice of  $n$  points, one from each pair  $\{x_i, y_i\}$ , form a circuit-hyperplane, then this is the rank  $n$  *free* spike; other spikes have such circuit-hyperplanes. A rank  $n$  *swirl* is obtained by adding a point freely to each 3-point line of the rank  $n$  whirl, then deleting those points lying on the intersection of two 3-point lines. If no  $n$  points form a circuit-hyperplane, this is the rank  $n$  *free* swirl; other swirls have such circuit hyperplanes (which necessarily have exactly one point from each of the original  $n$  lines used to construct the swirl).

In fact, as observed by Zaslavsky [11], spikes are lift matroids and swirls are frame matroids both coming from biased graphs of the form  $(2C_n, \mathcal{B})$  where every cycle in  $\mathcal{B}$  is of length  $n$ . In particular, the free spike of rank  $n$  is  $L(2C_n, \emptyset)$  and the free swirl of rank  $n$  is  $F(2C_n, \emptyset)$ . The family of biased graphs  $(2C_n, \mathcal{B}_n)$  defined above yields both an infinite antichain of spikes and swirls, since in both cases these matroids have exactly two circuit hyperplanes which partition the ground set, but the same is not true of any proper minor.

For every group  $\Gamma$ , we let  $\mathcal{F}_\Gamma$  (resp.  $\mathcal{L}_\Gamma$ ) denote the class of matroids which can be represented as a frame (lift) matroid of a biased graph which is  $\Gamma$ -labellable. It is straightforward to check that each of these classes is minor closed (this argument will be sketched later). In general, a matroid in either of these classes may have many different representations as biased graphs, and this complicates the problem of determining the excluded minors. Fortunately, our constructions have essentially unique representations, and this permits us to achieve the following somewhat surprising result (which we prove in Section 5).

**Theorem 1.6.** *For every infinite group  $\Gamma$  and every  $t \geq 3$  the classes  $\mathcal{L}_\Gamma$  and  $\mathcal{F}_\Gamma$  have infinitely many excluded minors of rank  $t$ .*

In addition, we prove that for every infinite group  $\Gamma$  and every  $t \geq 3$  there exist infinite antichains of rank  $t$  matroids within both  $\mathcal{L}_\Gamma$  and  $\mathcal{F}_\Gamma$ .

## 2 A Topological Characterisation

Theorem 1.1 consists of statements 1 and 3 of Theorem 2.1, which we prove next. For a graph  $G$ , group labelled by  $\phi : E(G) \rightarrow \Gamma$ , our basic definitions assign a notion of balance to each cycle. This notion naturally extends from cycles to closed walks. For an arbitrary closed walk  $W$ , we define  $W$  to be *balanced* if  $\phi(W) = 1$  and call it *unbalanced* otherwise.

Let  $W$  be a closed walk in the biased graph  $(G, \mathcal{B})$ , let  $W'$  be a subwalk of  $W$  which is a path from  $u$  to  $v$  and assume that  $C$  is a balanced cycle of  $G$  which contains the path  $W'$ . Let  $W''$  be the path from  $u$  to  $v$  in  $C$  distinct from  $W'$  and modify  $W$  to a new closed walk  $W^*$  by replacing  $W'$  by  $W''$ . In this case we say that  $W^*$  is obtained from  $W$  by *rerouting along a balanced cycle*, or simply, by a *balanced rerouting*. If  $\mathcal{B} = \mathcal{B}_\phi$  for a group labelling  $\phi$ , then since  $C$  is balanced,  $\phi(W') = \phi(W'')$ , so  $\phi(W^*) = \phi(W)$ .

**Theorem 2.1.** *Let  $(G, \mathcal{B})$  be a biased graph and let  $K$  be the 2-cell complex obtained from  $G$  by adding a disc with boundary  $C$  for every  $C \in \mathcal{B}$ . Then the following are equivalent.*

1.  $G$  is group labellable
2.  $G$  is  $\pi_1(K)$ -labellable.
3. Every cycle  $C \notin \mathcal{B}$  is noncontractible in  $K$ .

4. *There does not exist a sequence of closed walks  $W_1, \dots, W_n$  so that each  $W_{i+1}$  is obtained from  $W_i$  by a balanced rerouting,  $W_1$  is a simple walk around an unbalanced cycle and  $W_n$  is a simple walk around a balanced cycle.*

*Proof.* Trivially (2) implies (1), and our preceding discussion noted that (1) implies (4). So, to complete the proof it will suffice to show that (3) implies (2), and the negation of (3) implies the negation of (4).

We may assume that  $G$  is a connected graph (as the theorem operates independently on components) and choose a spanning tree  $T$ . Let  $(G', \mathcal{B}')$  denote the (one vertex) biased graph obtained from  $(G, \mathcal{B})$  by contracting every edge in  $E(T)$ . Let  $K'$  denote the cell complex obtained from  $K$  by identifying  $T$  to a single point. Since  $T$  is contractible, it follows that  $\pi_1(K) \cong \pi_1(K')$  (see Proposition 0.17 in [3]).

We now apply a standard result to obtain a natural description of the fundamental group of  $K'$ . Give  $G'$  an arbitrary orientation, and for every edge  $e \in E(G')$  let  $\gamma_e$  be a variable. For every cycle  $C \in \mathcal{B}$  choose a simple closed walk around  $C$ , and let  $e_1, \dots, e_m$  be the sequence of edges of this walk appearing in  $E(G')$  (so this closed walk becomes a sequence of loops on the single vertex of  $G'$ , obtained by removing from the closed walk around  $C$  those edges in  $T$ ). For  $i \in \{1, \dots, m\}$ , define  $\epsilon_i$  to be 1 if  $e_i$  is forward in this walk and  $-1$  if it is traversed backward. Now define  $\beta_C$  to be the word  $\gamma_{e_1}^{\epsilon_1} \gamma_{e_2}^{\epsilon_2} \dots \gamma_{e_m}^{\epsilon_m}$ . Define  $\Gamma$  to be the group presented by the generating set  $\{\gamma_e \mid e \in E(G')\}$  with the relations given by setting the words in  $\{\beta_C \mid C \in \mathcal{B}\}$  to be the identity. It follows from an application of Van Kampen's Theorem (see Section 1.2 in [3]) that  $\Gamma \cong \pi_1(K') \cong \pi_1(K)$  and furthermore, a closed walk  $W$  given by the edge sequence  $e_1, \dots, e_m$  with orientations  $\epsilon_1, \dots, \epsilon_m$  will be contractible in  $K'$  if and only if the product  $\prod_{i=1}^m \gamma_{e_i}^{\epsilon_i}$  is equal to the identity in  $\Gamma$ .

Our next step will be to define a  $\Gamma$ -labelling of the graph  $G$  given by  $\phi : E(G) \rightarrow \Gamma$ . For an edge  $e \in E(T)$ , we orient it arbitrarily and assign  $\phi(e) = 1$ . For an edge  $e \in E(G) \setminus E(T)$  we orient  $e$  as it was oriented in  $G'$  and then define  $\phi(e) = \gamma_e$ . Let  $W$  be a closed walk in  $G$  and let  $W'$  be the corresponding closed walk in  $G'$ . Suppose that  $W'$  has edge sequence

$e_1, \dots, e_m$  and that  $\epsilon_i = 1$  if  $e_i$  is forward in  $W'$  and  $\epsilon_i = -1$  if it is backward. Now we have

$$W \text{ is contractible in } K \Leftrightarrow W' \text{ is contractible in } K'$$

$$\begin{aligned} &\Leftrightarrow \prod_{i=1}^m \gamma_{e_i}^{\epsilon_i} = 1 \\ &\Leftrightarrow \phi(W) = 1 \end{aligned}$$

Every balanced cycle in  $G$  will be contractible in  $K$ , so we automatically have  $\mathcal{B} \subseteq \mathcal{B}_\phi$ . If (3) holds, then every cycle  $C \notin \mathcal{B}$  is uncontractible in  $K$  and the above equation implies that  $\mathcal{B} = \mathcal{B}_\phi$  so  $(G, \mathcal{B})$  is  $\Gamma$ -labellable and (2) holds. On the other hand, if (3) is violated, there is a cycle  $C \notin \mathcal{B}$  which is contractible in  $K$ , and a simple closed walk  $W_1$  around  $C$  will satisfy  $\phi(W_1) = 1$ . In this case, the group relations in  $\Gamma$  which reduce the product of the corresponding edge labels to the identity yield a sequence of closed walks which violate (4).  $\square$

In the preceding theorem it is shown that whenever  $(G, \mathcal{B})$  has a group labelling, it has one using the group  $\pi_1(K)$ . In fact, the labelling using this group constructed in the proof has a natural extreme property. If  $\phi$  and  $\psi$  are two group labellings of  $(G, \mathcal{B})$ , then by definition we have  $\mathcal{B}_\phi = \mathcal{B} = \mathcal{B}_\psi$  so these group labellings have the same set of balanced cycles. However, it is quite possible for a closed walk  $W$  to satisfy  $\phi(W) = 1$  and  $\psi(W) \neq 1$ . The group labelling constructed in the above proof has the unique minimal set of balanced closed walks. That is, any closed walk which is balanced in the group-labelling defined there will also be balanced under any other valid group-labelling.

### 3 General Construction

In this section we will utilize Theorem 1.1 to give a general construction of some biased graphs which are minor minimally not group labellable.

**Construction:** Let  $G$  be a simple graph embedded in the plane which is equipped with a  $t$ -vertex colouring satisfying the following:

1.  $G$  is a subdivision of a 3-connected graph.



2. Every colour appears exactly once on every face (so every face has size  $t$ ).
3. Every cycle of  $G$  of size  $\leq t$  is the boundary of a face.

Now we form a graph  $\tilde{G}$  from  $G$  by identifying each colour class to a single vertex. Define  $\mathcal{B}$  to be the set of all cycles of  $\tilde{G}$  which correspond to boundaries of finite faces of  $G$ .

We claim that  $(\tilde{G}, \mathcal{B})$  is a biased graph. Since every cycle in  $\mathcal{B}$  is a Hamiltonian cycle of  $\tilde{G}$ , the only way for a theta subgraph of  $\tilde{G}$  to contain two members of  $\mathcal{B}$  would be for this theta subgraph to have two edges in parallel. However this contradicts the assumption that  $G$  is simple. Thus each theta subgraph of  $\tilde{G}$  contains at most one member of  $\mathcal{B}$  and we conclude that  $(\tilde{G}, \mathcal{B})$  is a biased graph.

**Theorem 3.1.** *The biased graph  $(\tilde{G}, \mathcal{B})$  constructed above is not group labellable. For every edge  $e$  and every infinite group  $\Gamma$ , each of the biased graphs obtained by deleting and contracting  $e$  are  $\Gamma$ -labellable.*

*Proof.* Let  $K$  be the 2-cell complex obtained from the embedded graph  $G$  by removing the infinite face. Thus  $K$  is a disc and its boundary is a cycle  $C$ . Now let  $\tilde{K}$  be the 2-cell complex obtained from  $K$  by identifying each colour class of vertices to a single point. The cycle  $C$  is a contractible curve in  $K$ , so it is also a contractible curve in  $\tilde{K}$ . Since  $C \notin \mathcal{B}$ , by Theorem 1.1,  $(\tilde{G}, \mathcal{B})$  is not group-labellable.

For the second part of the proof construct group labellings for which only certain controlled group products give the identity. In preparation for this we now choose a useful sequence of group elements. Let  $\Gamma$  be an arbitrary infinite group (written multiplicatively) and choose  $g_0 \in \Gamma \setminus \{1\}$ . Now for  $1 \leq k \leq |E(G)| + |V(G)|$  choose  $g_k \in \Gamma$  so that  $g_k$  cannot be expressed as a word of length  $\leq 3t$  using  $g_0, g_0^{-1}, \dots, g_{k-1}, g_{k-1}^{-1}$ . Let  $e \in E(G)$ . We consider the two cases contraction and deletion of  $e$ .

#### *Contraction*

Consider the biased graph  $(\tilde{G}', \mathcal{B}')$  obtained from  $(\tilde{G}, \mathcal{B})$  by contracting edge  $e$ . Since every cycle in  $\mathcal{B}$  is Hamiltonian in  $\tilde{G}$ , every such cycle not containing  $e$  will form handcuffs upon contracting  $e$ . So the only cycles in  $\mathcal{B}'$  correspond to finite faces of the planar graph  $G$  which contain  $e$  (and thus  $|\mathcal{B}'| \leq 2$ ). To group label  $\tilde{G}'$ , we label  $E(G) \setminus e$ ;  $\tilde{G}'$  then inherits its labels from  $G/e$ . Let  $H$  be the subgraph of  $G$  consisting of all of the vertices and edges which are on a finite face containing  $e$ . It follows from the assumption that  $G$

is a subdivision of a 3-connected graph that  $H$  must either be a cycle or a theta subgraph (depending on whether  $e$  lies on the infinite face or not). Let  $V(H/e) = \{v_0, \dots, v_n\}$  and let  $E(G) \setminus E(H) = \{e_{n+1}, \dots, e_m\}$ . Now to construct the group labelling, we give  $G$  an arbitrary orientation, and we assign edge labels as follows. For every edge  $f \in E(H/e)$ , if  $f = v_i v_j$ , oriented from  $v_i$  to  $v_j$ , let  $\phi(f) = g_i^{-1} g_j$ . For every edge  $e_k \in E(G) \setminus E(H)$ , define  $\phi(e_k) = g_k$ .

We claim that  $\phi$  is a group-labelling of  $(\tilde{G}', \mathcal{B}')$ ; *i.e.* that  $\mathcal{B}_\phi = \mathcal{B}'$ . To prove this, let  $\tilde{D}$  be an arbitrary cycle in  $\tilde{G}'$ . We show that either  $\tilde{D}$  is in both  $\mathcal{B}'$  and  $\mathcal{B}_\phi$  or  $\tilde{D}$  is in neither. Define  $D$  to be the subgraph of  $G$  induced by  $E(\tilde{D})$  (so  $D$  is either a cycle or a union of disjoint paths). First suppose that  $\tilde{D}$  contains an edge  $e_k \in E(G) \setminus E(H)$ , and choose such an edge for which  $k$  is maximum. Since  $e_k \in E(\tilde{D})$  we have  $\tilde{D} \notin \mathcal{B}'$ . If  $W$  is a simple closed walk in  $\tilde{G}'$  around  $\tilde{D}$  beginning with  $e_k$  in the forward direction, then  $\phi(W)$  has the form  $g_k$  times a word of length  $< 2(t-1) < 3t$  consisting of group elements in  $\{g_0, g_0^{-1}, \dots, g_{k-1}, g_{k-1}^{-1}\}$ . Thus  $\phi(W) \neq 1$  and we have  $\tilde{D} \notin \mathcal{B}_\phi$  as desired. So now suppose  $E(\tilde{D}) \subseteq E(H)$ . If  $D$  is a cycle in  $H/e$ , then  $\tilde{D} \in \mathcal{B}'$  and  $\tilde{D} \in \mathcal{B}_\phi$  by definition. If  $D$  is not a cycle in  $H/e$ , then  $\tilde{D} \notin \mathcal{B}'$  and we must show that  $\tilde{D} \notin \mathcal{B}_\phi$ . Let  $D_1, \dots, D_r$  be the components of  $D$ , let  $W$  be a simple closed walk around  $\tilde{D}$  and assume that  $W$  encounters the edges of each  $D_i$  consecutively. If the subwalk  $W'$  of  $W$  traversing  $D_h$  begins at  $v_i$  and ends at  $v_j$ , then we have  $\phi(W') = g_i^{-1} g_j$ . Therefore, if we choose  $k$  to be the largest value so that  $v_k$  is an endpoint of one of the paths  $D_1, \dots, D_r$  then  $\phi(W)$  may be expressed as a word of length  $\leq 2r < 2(t-1) < 3t$  using exactly one copy of  $g_k$  or  $g_k^{-1}$  with all other terms equal to one of  $g_0, g_0^{-1}, \dots, g_{k-1}, g_{k-1}^{-1}$ . It follows that  $\tilde{D} \notin \mathcal{B}_\phi$  as desired.

### Deletion

Now consider the biased graph  $(\tilde{G}', \mathcal{B}')$  obtained from  $(\tilde{G}, \mathcal{B})$  by deleting  $e$ . First suppose that  $e$  is incident with the infinite face of  $G$ . In this case, let  $V(G) = \{v_0, \dots, v_n\}$  and associate each  $v_i$  with the group element  $g_i$ . Orient the edges in  $E \setminus e$  arbitrarily, and for every  $f \in E \setminus e$  oriented from  $v_i$  to  $v_j$  define  $\phi(f) = g_i^{-1} g_j$ . We claim that  $\mathcal{B}_\phi = \mathcal{B}'$ . To prove this (as before) we let  $\tilde{D}$  be an arbitrary cycle in  $\tilde{G}'$  and we let  $D$  be the corresponding subgraph of  $G \setminus e$ . As before, the graph  $D$  must either be a cycle or a union of disjoint paths. If  $D$  is a cycle, then by the third property, it must be a face boundary, so  $\tilde{D} \in \mathcal{B}'$  by definition and  $\tilde{D} \in \mathcal{B}_\phi$  by construction. If  $D$  is a union of disjoint paths given by  $D_1, \dots, D_r$ , then  $\tilde{D} \notin \mathcal{B}'$  and we must show that  $\tilde{D} \notin \mathcal{B}_\phi$ . As before, choose a closed walk  $W$  traversing  $\tilde{D}$  so that it encounters the edges of each  $D_h$  consecutively. If the subwalk  $W'$  of  $W$  traversing

$D_h$  starts at  $v_i$  and ends at  $v_j$  our definitions imply  $\phi(W') = g_i g_j^{-1}$ . So, as before, if  $k$  is the largest value so that  $v_k$  is an endpoint of one of the paths  $D_1, \dots, D_r$ , we find that  $\phi(W)$  may be written as a word of length  $\leq 2r \leq 2t < 3t$  using one copy of either  $g_k$  or  $g_k^{-1}$  with all other terms one of  $g_0, g_0^{-1}, \dots, g_{k-1}, g_{k-1}^{-1}$ . It follows that  $\tilde{D} \notin \mathcal{B}_\phi$  as desired.

Next suppose that  $e$  is not incident with the infinite face and let  $R$  be the new face in  $G \setminus e$  formed by deleting  $e$  from  $G$ . Choose a path  $P$  in the dual graph of  $G \setminus e$  from the infinite face to  $R$  and then orient the edges in  $E \setminus e$  so that the edges dual to those in  $P$  cross the path  $P$  consistently (for instance, if  $P$  is given a direction, then  $E \setminus e$  may be oriented so that each edge dual to one in  $P$  crosses  $P$  from the left to the right). Now let  $V(G) = \{v_0, \dots, v_n\}$  and define a group labelling as follows. If  $f$  is an edge from  $v_i$  to  $v_j$  and  $f$  is not dual to one in  $P$  we let  $\phi(e) = g_i^{-1} g_j$ ; if  $e$  is dual to an edge in  $P$ , let  $\phi(e) = g_i^{-1} g_0 g_j$ . Observe that for any closed walk  $W$  in  $G \setminus e$  we have  $\phi(W) = g_0^s$  where  $s$  is the number of times the curve  $W$  winds around the face  $R$ . Following our above procedure, now let  $\tilde{D}$  be a cycle of  $\tilde{G}'$  and let  $D$  be the corresponding subgraph of  $G \setminus e$ . If  $D$  is a cycle, then since its length is at most  $t$ , it bounds a face in  $G \setminus e$  other than  $R$ . If this is a finite face, then  $\tilde{D} \in \mathcal{B}'$  and by definition  $\tilde{D} \in \mathcal{B}_\phi$ . If this is the infinite face, then  $\tilde{D} \notin \mathcal{B}'$  and since this face winds around  $R$  exactly once we have  $\phi(W) = g_0$  or  $\phi(W) = g_0^{-1}$ , so  $\tilde{D} \notin \mathcal{B}_\phi$ . Finally, if  $D$  is a union of the disjoint paths  $D_1, \dots, D_r$  then  $\tilde{D} \notin \mathcal{B}'$  and we must show  $\tilde{D} \notin \mathcal{B}_\phi$ . Choose a closed walk  $W$  traversing  $\tilde{D}$  encountering the edges of each  $D_h$  consecutively. Let  $W = e_1 e_2 \dots e_s$ . Then  $s \leq t$ , and  $\phi(W) = \phi(e_1) \phi(e_2) \dots \phi(e_s)$  is a word of length  $\leq 3s$  since each word  $\phi(e_i)$  is a word of the form  $g_i^{-1} g_j$ ,  $g_i^{-1} g_0 g_j$ , or  $g_i^{-1} g_0^{-1} g_j$ , and so has length at most 3. Letting  $k$  be the largest value so that  $v_k$  is an endpoint of one of the paths  $D_1, \dots, D_r$  we have that  $\phi(W)$  may be written as a word of length  $\leq 3t$  using one copy of either  $g_k$  or  $g_k^{-1}$  and all other terms one of  $g_0, g_0^{-1}, \dots, g_{k-1}, g_{k-1}^{-1}$ . As before, this implies that  $\phi(W) \neq 1$  so  $\tilde{D} \notin \mathcal{B}_\phi$  as desired.  $\square$

## 4 Excluded Minors - Biased Graphs

In this section we will prove Theorem 1.3, giving us a large collection of minor-minimal not group labellable biased graphs each of whose underlying simple graph (*i.e.* ignoring parallel edges) is complete. We then construct some families of minor-minimal not group labellable biased graphs each of whose underlying simple graphs is a cycle. These results are based upon

the general construction from the previous section together with certain families of coloured planar graphs. We begin by introducing two basic families of coloured planar graphs. (These colourings are proper.)

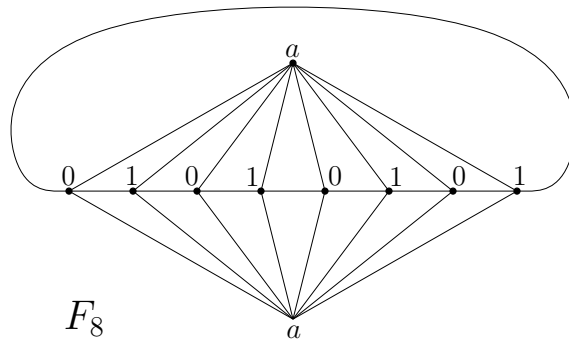


Figure 1

For every positive integer  $k$  we define  $F_{2k}$  to be the coloured planar graph given as follows. Begin with a cycle of length  $2k$  embedded in the plane in which vertices are alternately coloured 0 and 1. Then add two additional vertices, one in each face, each adjacent to all vertices on this cycle and each of colour  $a$  (Figure 1).

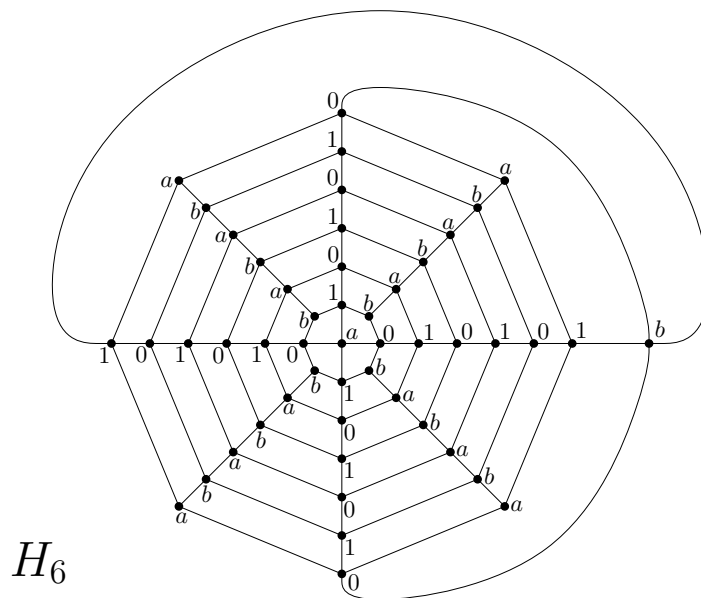


Figure 2

For every positive integer  $k$  we define  $H_{2k}$  to be the planar graph constructed as follows.

Begin with  $2k$  nested 8-cycles embedded in the plane, each joined to the previous and the next by a perfect matching. Colour this portion of the graph by colouring the innermost cycle  $b, 0, b, 1, b, 0, b, 1$  and extend this colouring so that every 4-cycle (of the present graph) contains exactly one vertex of each of the colours  $\{a, b, 0, 1\}$  (this extension is unique). Finally, add a vertex  $v_1$  in the inner 8-cycle of colour  $a$  joined to all vertices on this cycle not of colour  $b$  and similarly, add a vertex  $v_2$  in the infinite face coloured  $b$  and adjacent to all vertices not of colour  $a$  on this face (Figure 2). Next we use these to construct some useful families of coloured planar graphs.

**Lemma 4.1.** *For every  $t \geq 3$  and  $\ell$  there exists a  $t$ -coloured planar graph with the following properties:*

1.  *$G$  is a subdivision of a 3-connected graph.*
2. *Every colour appears exactly once on every face (so every face has size  $t$ ).*
3. *Every cycle of  $G$  of size  $\leq t$  is the boundary of a face.*
4. *Every pair of distinct colours appear on opposite ends of at least  $\ell$  edges.*

*Proof.* We split into cases depending on the parity of  $t$ .

*$t$  odd*

For  $t = 3$  the coloured graphs  $F_{2k}$  with  $k$  suitably large satisfy the constraints. In general, we choose  $s$  so that  $t = 2s + 1$ , use the colour set  $\{a\} \cup \{1, 2, \dots, 2s\}$ , and take  $k$  as large as necessary to achieve what is required in each step. Begin by choosing a sequence  $x_1, x_2, \dots, x_{2k}$  of elements from  $\{1, 2, \dots, 2s\}$  with the property that every  $x_i$  has the same parity as  $i$  and further, every pair of numbers in  $\{1, 2, \dots, 2s\}$  with differing parities appear consecutively in this sequence at least  $\ell$  times. Now modify the colouring of the graph  $F_{2k}$  by replacing the sequence of 0 and 1 colours by  $x_1, \dots, x_{2k}$ . Next, for every edge with one end of colour  $a$  and the other end an odd (even) colour  $i$  we subdivide this edge  $s - 1$  times and give these new vertices distinct odd (even) colours in  $\{1, \dots, 2s\} \setminus \{i\}$ . Let  $v_1, v_2$  be the two vertices of colour  $a$ . By carefully choosing the colours on the degree 2 vertices cofacial with  $v_1$  we may arrange that  $v_1$  has at least  $\ell$  neighbours of each colour  $\{1, \dots, 2s\}$ . By carefully choosing the colours on the degree 2 vertices cofacial with  $v_2$  we may arrange that

every pair of distinct colours of the same parity appear on opposite ends of at least  $\ell$  edges. The resulting coloured planar graph then has the desired properties.

*t even*

Note that for  $t = 4$  the coloured graphs  $H_{2k}$  with  $k$  suitably large satisfy the construction. In general, we choose  $s$  so that  $t = 2s + 2$ , use the colour set  $\{a, b\} \cup \{1, 2, \dots, 2s\}$ , and take  $k$  as large as necessary to achieve what is required in each step. Begin by choosing a sequence  $x_1, \dots, x_{2k}$  of elements from  $\{1, \dots, 2s\}$  with the property that every  $x_i$  has the same parity as  $i$  and further, every pair of numbers in  $\{1, \dots, 2s\}$  with differing parities appears consecutively in this sequence at least  $\ell$  times. Now consider the coloured graph  $H_{2k}$ . Let  $P_1, P_3$  be the paths of length  $2k - 1$  that are coloured alternately 0 and 1 beginning with a vertex incident to  $v_1$  coloured 1, and let  $P_2, P_4$  be the paths of length  $2k - 1$  that are coloured alternatively 0 and 1 beginning with a vertex coloured 0 incident to  $v_1$ . Modify the colouring of  $H_{2k}$  by replacing the colours along each of  $P_1$  and  $P_3$  with the sequence of colours  $x_1, \dots, x_{2k}$ , and replacing the colours along each of  $P_2$  and  $P_4$  with the sequence of colours  $x_2, \dots, x_{2k}, x_1$ . Note that in this manner we have replaced each vertex previously coloured 0 with an even colour, and each vertex previously coloured 1 with an odd colour. Now we modify the graph further by the following procedure. Aside from the four edges incident with the central vertex  $v_1$  (coloured  $a$ ) and the four edges incident with outer vertex  $v_2$  (coloured  $b$ ), for every other edge

- $ai$  or  $bi$  with  $i$  even: subdivide the edge  $s - 1$  times and give each new vertex a distinct even colour from  $\{1, \dots, 2s\} \setminus \{i\}$ ;
- $ai$  or  $bi$  with  $i$  odd: subdivide the edge  $s - 1$  times and give each new vertex a distinct odd colour from  $\{1, \dots, 2s\} \setminus \{i\}$ .

By carefully choosing these new colours for the vertices cofacial with a vertex of  $P_1$  we may arrange that there are at least  $\ell$  edges with one end of colour  $a$  (resp. colour  $b$ ) and the other of colour  $i$  for every even (odd)  $i \in \{1, \dots, 2s\}$ . Similarly, by choosing these new colours for the vertices cofacial with a vertex of  $P_2$  we may arrange that there are at least  $\ell$  edges with one end of colour  $a$  (resp.  $b$ ) and the other of colour  $i$  for every odd (even)  $i \in \{1, \dots, 2s\}$ . Finally, by carefully choosing these new colours for the vertices cofacial with a vertex of  $P_3$  or  $P_4$  we may arrange that every pair of integers in  $\{1, \dots, 2s\}$  of the same parity appear

on opposite ends of at least  $\ell$  edges. The resulting coloured graph now has the desired properties.  $\square$

With this, we can easily prove our main result for this section.

*Proof of Theorem 1.3:* This follows from Theorem 3.1 and Lemma 4.1.  $\square$

Our next theorem gives constructions for families of minor-minimal not group labellable biased graphs each of whose underlying simple graph is a cycle.

**Theorem 4.2.** *For every  $k \geq 2$  and for  $t = 3$  and every  $t \geq 5$  there exists a biased graph  $(G, \mathcal{B})$  with the following properties:*

- *The underlying simple graph of  $G$  is  $C_t$ .*
- *If  $u, v$  are adjacent vertices they are joined by exactly  $2k$  edges.*
- *$(G, \mathcal{B})$  is not group-labellable.*
- *For every infinite group  $\Gamma$ , every proper minor of  $(G, \mathcal{B})$  is  $\Gamma$ -labellable.*

*Proof.* As in the previous theorem we will construct certain coloured planar graphs and then call upon Theorem 3.1. The graphs we construct have  $t$ -colourings using the colours  $\{0, 1, \dots, t-1\}$  with the following properties:

- On each face the cyclic ordering of colours is given by either  $0, 1, \dots, t-1$  or its reverse.
- There are exactly  $4k$  faces.

Note that the above two properties guarantee that the graph  $\tilde{G}$  obtained from the identification process in our construction will satisfy the first and second properties of the theorem. When  $t = 3$  we may obtain such a graph from  $F_{2k}$  by changing the two vertices coloured  $a$  to colour 2. So, we may assume  $t \geq 5$ . When  $(t, k) \in \{(5, 2), (8, 2)\}$  the graphs depicted in Figure 3 satisfy the desired properties.

Thus, we may assume  $(t, k) \notin \{(5, 2), (8, 2)\}$ . Let  $t = 3s + p + q$  where  $0 \leq p \leq q \leq 1$ . Modify  $F_{2k}$  by changing every vertex of colour  $a$  to colour  $s + p$  and every vertex of colour 1 to colour  $2s + p + q$ . Now subdivide every edge with ends of colours 0 and  $s + p$  exactly

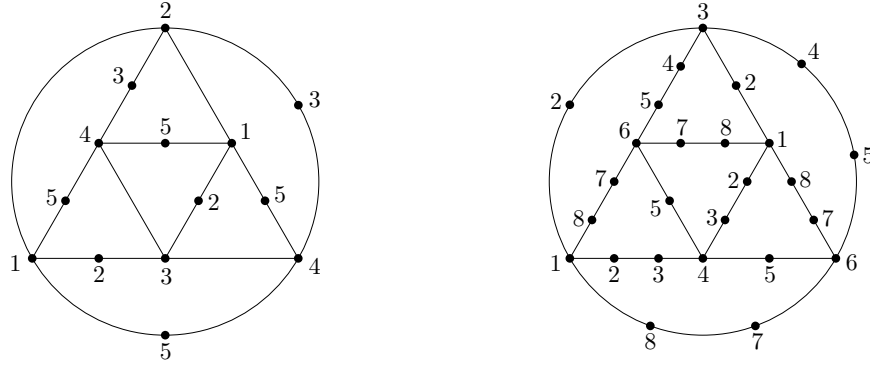


Figure 3

$s + p - 1$  times, every edge with ends of colours  $s + p$  and  $2s + p + q$  exactly  $s + q - 1$  times and every edge with ends of colours 0 and  $2s + p + q$  exactly  $s - 1$  times. Now we may colour the vertices of degree two so that around every face they are ordered cyclicly in either clockwise or counterclockwise direction as  $0, 1, \dots, 3s + p + q - 1$ .

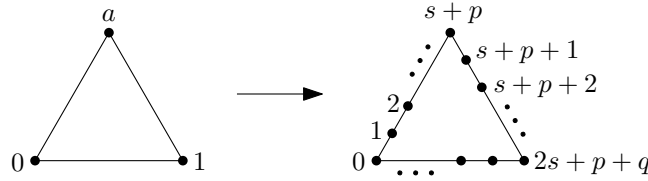


Figure 4

In this case each triangle is subdivided as in Figure 4. It follows immediately from our construction that the graphs we have constructed are subdivisions of 3-connected planar graphs with exactly  $4k$  faces all coloured as in the figure. So to complete the proof, we need only verify that these graphs have the property that every cycle of size  $\leq t$  is the boundary of a face. To verify this, observe that every such cycle contains at least four vertices of degree  $\geq 3$  so will have total length at least  $4\lfloor \frac{t}{3} \rfloor$  which is greater than  $t$  for all  $t \geq 6$  except for  $t = 8$ . In the cases when  $t = 5$  and  $t = 8$  we have the additional assumption  $k \geq 3$  so again here we have the desired property.  $\square$



## 5 Excluded Minors - Matroids

In this section we will call upon our prior results to construct some excluded minors for families of frame and lift matroids. If  $(G, \mathcal{B})$  is a biased graph, then we let  $L(G, \mathcal{B})$  (resp.  $F(G, \mathcal{B})$ ) denote the associated lift (frame) matroid. The minor operations we have defined for biased graphs commute with these operators  $L$  and  $F$  (for instance,  $L((G, \mathcal{B}) \setminus e) = L(G, \mathcal{B}) \setminus e$ ). However, we have not defined contraction of unbalanced loop edges in the setting of biased graphs. If  $e$  is a loop edge of the biased graph  $(G, \mathcal{B})$  which forms a balanced cycle, then  $e$  is a loop of both matroids  $L(G, \mathcal{B})$  and  $F(G, \mathcal{B})$  so contracting this loop element in the matroid is the same as deleting it and this new matroid is given (in both cases) by the biased graph  $(G, \mathcal{B}) \setminus e$ . On the other hand, if  $e$  is not a balanced cycle, then it is not a loop in either matroid. In this case the matroid  $L(G, \mathcal{B})/e$  is the lift matroid of the biased graph with underlying graph  $G \setminus e$  and with all cycles balanced. The matroid  $F(G, \mathcal{B})/e$  is the frame matroid of the biased graph  $(G', \mathcal{B}')$  which is obtained from  $(G, \mathcal{B})$  by the following “roll-up” procedure. If  $e$  is incident to vertex  $u$ , then  $G'$  is obtained from  $G$  by deleting  $e$  and  $u$ , and redefining the endpoints of each remaining edge  $f$  incident to  $u$  as follows. If  $f$  is a loop (balanced or unbalanced),  $f$  becomes a balanced loop incident to any neighbour of  $u$ ; if  $f = uv$  with  $v \neq u$ ,  $f$  becomes an unbalanced loop incident to  $v$ . The set of balanced cycles  $\mathcal{B}'$  is the set  $\{C \in \mathcal{B} : u \notin V(C)\}$ .

It follows from the above discussion that the classes of both frame and lift matroids are closed under minors. Next fix a group  $\Gamma$  and consider the classes  $\mathcal{L}_\Gamma$  and  $\mathcal{F}_\Gamma$  of lift and frame matroids of biased graphs which are  $\Gamma$ -labellable. We wish to show that these classes are also closed under minors. Since we have already shown that  $\Gamma$ -labellable biased graphs are closed under (biased graph) minor operations, we need only consider the effect of contracting an element  $e$  in one of these matroids which corresponds to an unbalanced loop edge in the graph. In the lift matroid, contracting  $e$  gives us a new lift matroid of a graph with all cycles balanced. Since this biased graph is clearly  $\Gamma$ -labellable we have that  $\mathcal{L}_\Gamma$  is minor closed. Next suppose  $\phi$  is a  $\Gamma$ -labelling of  $(G, \mathcal{B})$  and  $e$  is a loop edge with  $\phi(e) \neq 1$ . In this case, the biased graph obtained by the above roll-up procedure will again be group-labellable; we may obtain such a labelling from  $\phi$  by setting  $\phi(f) = 1$  for any other loop edge  $f$  with the same ends as  $e$  and setting  $\phi(f) \neq 1$  for any edge  $f$  which was rolled up. We conclude that  $\mathcal{F}_\Gamma$  is also closed under minors.

We have already shown numerous families of biased graphs which are minor minimal subject to not being  $\Gamma$ -labellable. However, this does not immediately give us excluded minors for the classes  $\mathcal{L}_\Gamma$  and  $\mathcal{F}_\Gamma$  since there might exist two biased graphs with the same frame (lift) matroid where one is  $\Gamma$ -labellable and the other is not. In order to show that we do have excluded minors we will need to handle this issue of non-unique representations.

To assist in this exploration, we begin by looking at biased graphs which have associated matroids isomorphic to  $U_{2,m}$ . Define  $K_2^m$  to be a two vertex graph consisting of  $m$  edges in parallel, let  $K_2^{m+}$  be a graph obtained from  $K_2^m$  by adding a single loop edge, and let  $K_2^{m++}$  be a graph obtained from  $K_2^{m+}$  by adding another loop not adjacent to the first loop.

**Observation 5.1.** *For a biased graph  $(G, \mathcal{B})$  and  $m \geq 4$  we have:*

1.  $L(G, \mathcal{B}) \cong U_{2,m}$  if and only if  $(G, \mathcal{B})$  is isomorphic to  $(K_2^m, \emptyset)$  or  $(K_2^{(m-1)+}, \emptyset)$ .
2.  $F(G, \mathcal{B}) \cong U_{2,m}$  if and only if  $(G, \mathcal{B})$  is isomorphic to  $(K_2^m, \emptyset)$ ,  $(K_2^{(m-1)+}, \emptyset)$ , or  $K_2^{(m-2)++}, \emptyset)$ .

Let us declare that a biased graph  $(G, \mathcal{B})$  is *lift-unique* (*frame-unique*) if the only biased graphs with lift (frame) matroid isomorphic to  $L(G, \mathcal{B})$  ( $F(G, \mathcal{B})$ ) are obtained from  $(G, \mathcal{B})$  by renaming the vertices.

**Lemma 5.2.** *Let  $(G, \mathcal{B})$  be a loopless biased graph on  $n \geq 3$  vertices for which every pair of vertices are joined by at least four edges, and all cycles of length two are unbalanced. Then  $(G, \mathcal{B})$  is both frame-unique and lift-unique.*

*Proof.* We begin by considering  $F(G, \mathcal{B})$ . Let  $E = E(G)$  and define a relation  $\sim$  on  $E$  by the rule that  $e \sim f$  if there exists a restriction of the frame matroid isomorphic to  $U_{2,4}$  which contains both  $e$  and  $f$ . It follows easily from the description of  $(G, \mathcal{B})$  that  $\sim$  is an equivalence relation and its equivalence classes are precisely the parallel classes of  $G$ , which we denote by  $E_1, E_2, \dots, E_{\binom{n}{2}}$ .

Suppose that  $(G', \mathcal{B}')$  is another biased graph on the same edge set with the same frame matroid; i.e.,  $F(G', \mathcal{B}') \cong F(G, \mathcal{B})$ . If  $|E_i| = m$  then the restriction of our matroid to  $E_i$  is isomorphic to  $U_{2,m}$  and thus in the graph  $G'$ , the edges in  $E_i$  induce a two vertex subgraph isomorphic to one of  $K_2^m$ ,  $K_2^{(m-1)+}$  or  $K_2^{(m-2)++}$ . It follows from the fact that  $\sim$  is an equivalence relation that for  $i \neq j$  the edge sets  $E_i$  and  $E_j$  induce graphs on distinct two

vertex sets. Next suppose (for a contradiction) that there exists a loop edge  $e$  in  $G'$  incident with the vertex  $v$ . Let  $e', e''$  be non-loop edges incident with  $v$  which are not in parallel. Then  $e' \in E_i$  and  $e'' \in E_j$  for some  $i \neq j$ . Therefore  $e$  is in both  $E_i$  and  $E_j$ , a contradiction. Thus the graph  $G'$  is loopless, and  $E_1, \dots, E_{\binom{n}{2}}$  are also its parallel classes.

Let  $e, f, g$  be three edges which form a triangle in  $G$  and let  $e'$  be parallel with  $e$ . Then one of  $\{e, f, g\}$ ,  $\{e', f, g\}$ ,  $\{e, e', f, g\}$  is a circuit in  $F(G, \mathcal{B})$ . It follows from this, and the fact that  $G'$  is loopless with the same parallel classes as  $G$ , that the edges  $e, f, g$  must also form a triangle in  $G'$ . In particular, this implies that two edges  $e, f$  are adjacent in  $G$  if and only if they are adjacent in  $G'$ .

Therefore, the line graphs of  $G$  and  $G'$  are isomorphic. For  $n \geq 5$  the maximum cliques in the line graph of  $K_n$  correspond precisely to sets of edges incident with a common vertex, and it follows that for  $n \geq 5$  the biased graph  $(G', \mathcal{B}')$  may be obtained from  $(G, \mathcal{B})$  by renaming the vertices. For  $n = 3$  there is also nothing left to prove, so we are left with the case  $n = 4$ . The maximum cliques of the line graph of  $K_4$  are given by either triangles or sets of edges incident with a common vertex. Since three edges form a triangle in  $G$  if and only if they form a triangle in  $G'$  we conclude that again in this case, the biased graph  $(G', \mathcal{B}')$  may be obtained from  $(G, \mathcal{B})$  by renaming the vertices. We conclude that  $(G, \mathcal{B})$  is frame-unique.

For lift matroids the same proof applies with the only difference being that the two vertex subgraph induced by  $E_i$  cannot be  $K_2^{(m-2)++}$ .  $\square$

*Proof of Theorem 1.6:* Let  $t \geq 3$  and let  $\Gamma$  be an infinite group. By Theorem 1.3 we may choose an infinite set of biased graphs on  $t$  vertices  $\{(G_i, \mathcal{B}_i) \mid i \in \{1, 2, \dots\}\}$  with  $|E(G_{i+1})| > |E(G_i)|$  so that every pair of vertices is joined by at least 4 edges in every  $G_i$ , every  $(G_i, \mathcal{B}_i)$  is not  $\Gamma$ -labellable, and every proper minor of  $(G_i, \mathcal{B}_i)$  is  $\Gamma$ -labellable. By the previous lemma, each  $(G_i, \mathcal{B}_i)$  is both frame-unique and lift-unique. We conclude  $F(G_i, \mathcal{B}_i) \notin \mathcal{F}_\Gamma$  and  $L(G_i, \mathcal{B}_i) \notin \mathcal{L}_\Gamma$ . Since  $\text{rank}(F(G, \mathcal{B})) = \text{rank}(L(G, \mathcal{B})) = |V(G)|$ , it follows that the lift (resp. frame) matroid of every  $(G_i, \mathcal{B}_i)$  is an excluded minor of rank  $t$  for the class  $\mathcal{L}_\Gamma$  ( $\mathcal{F}_\Gamma$ ).  $\square$

## 6 Infinite Antichains

In the world of biased graphs, we have now constructed infinite antichains of bounded number of vertices by finding biased graphs which are minor minimal subject to not being group labellable. In this section we will prove that there also exist infinite antichains of bounded number of vertices within the family  $\mathcal{G}_\Gamma$  for every infinite group  $\Gamma$ . We shall also show that these results extend to give us infinite antichains of bounded rank within the matroid families  $\mathcal{L}_\Gamma$  and  $\mathcal{F}_\Gamma$ . We begin by noting that the biased graphs  $(2C_n, \mathcal{B}_n)$  from the introduction are group labellable (recall that in this case  $\mathcal{B}_n$  consists of two edge disjoint cycles of length  $n$ ).

**Observation 6.1.** *For every infinite group  $\Gamma$  and every  $n \geq 2$  the biased graph  $(2C_n, \mathcal{B}_n)$  is  $\Gamma$ -labellable.*

*Proof.* Orient the edges so that each of the two balanced cycles is a directed cycle, and label all edges in the first balanced cycle with 1. Let  $e_1, \dots, e_n$  be the edges of the second balanced cycle in order. Now choose a sequence of group elements  $g_1, \dots, g_{n-1}$  so that no subsequence of these elements has product equal to 1 (this may be done greedily). Assign  $e_i$  the label  $g_i$  for  $1 \leq i \leq n-1$  and assign  $e_n$  the label  $g_{n-1}^{-1} \dots g_1^{-1}$ . It follows easily that  $(2C_n, \mathcal{B}_n)$  is the biased graph associated with this group labelling, as desired.  $\square$

So, for every infinite group  $\Gamma$  we have an infinite antichain of biased graphs within  $\mathcal{G}_\Gamma$ . Next we will show (using an argument very similar to that in the proof of Theorem 3.1) that we can find such antichains with a bounded number of vertices.

**Lemma 6.2.** *For every infinite group  $\Gamma$  and every  $t \geq 3$  there exists an infinite antichain in  $\mathcal{G}_\Gamma$  of biased graphs on  $t$  vertices.*

*Proof.* Apply Lemma 4.1 to choose an infinite family of  $t$ -coloured planar graphs  $\{G_1, G_2, \dots\}$  each of which has a distinct number of edges. Now for every  $k$ , let  $\widetilde{G}_k$  be the graph obtained from  $G_k$  by identifying each colour class to a single vertex. Define  $\mathcal{B}_k$  to be the set of cycles which are faces of the planar embedding of  $G_k$ .

First we will prove that every  $(\widetilde{G}_k, \mathcal{B}_k)$  is in  $\mathcal{G}_\Gamma$ . To this end, we let  $V(G) = \{v_1, \dots, v_n\}$  and choose a sequence of group elements  $g_1, \dots, g_n$  with the property that each  $g_i$  cannot be represented as a product of distinct elements from the set  $\{g_1, g_1^{-1}, \dots, g_i, g_i^{-1}\}$  (in any order). Now orient the edges of  $G_k$  and thus  $\widetilde{G}_k$  arbitrarily, and for every edge  $e$  from  $v_i$  to  $v_j$  define

$\phi(e) = g_i^{-1}g_j$ . We claim that  $\mathcal{B}_\phi = \mathcal{B}_k$ . To see this, let  $C$  be an arbitrary cycle in  $\widetilde{G}_k$ . If  $C$  is also a cycle in  $G_k$ , then it must bound a face, so we have  $C \in \mathcal{B}_k$  and, by our construction  $C \in \mathcal{B}_\phi$ . Otherwise, the set of edges  $E(C)$  forms a collection of paths in  $G_k$ , say  $D_1, \dots, D_r$ . Now choose a closed walk  $W$  around the cycle  $C$  in  $\widetilde{G}_k$  and assume that  $W$  encounters the edges of each  $D_i$  consecutively. It now follows from our construction that  $\phi(W)$  may be expressed as a product of distinct group elements from  $S = \{g_i \mid v_i \text{ is an end of some } D_j\}$  together with  $S^{-1}$ . It now follows from our choice of group elements that this product is not the identity. So, in this case we have that  $C \notin \mathcal{B}_k$  and  $C \notin \mathcal{B}_\phi$  as desired.

It remains to prove that  $\{(\widetilde{G}_k, \mathcal{B}_k) \mid k \in \mathbb{N}\}$  is an antichain. Suppose (for a contradiction) that  $(\widetilde{G}_i, \mathcal{B}_i)$  contains a biased graph isomorphic to  $(\widetilde{G}_j, \mathcal{B}_j)$  as a minor. Since these graphs have the same number of vertices, it must be that  $(\widetilde{G}_j, \mathcal{B}_j)$  is isomorphic to  $(\widetilde{G}_i, \mathcal{B}_i) \setminus S$  for some nonempty set of edges  $S$ . Now choose an edge  $e \in E(\widetilde{G}) \setminus S$  which lies on a common face with an edge in  $S$ . The edge  $e$  will be in at most one balanced cycle in  $(\widetilde{G}_i, \mathcal{B}_i) \setminus S$ , but every edge in  $(\widetilde{G}_j, \mathcal{B}_j)$  is contained in exactly two balanced cycles. This contradiction shows that  $\{(\widetilde{G}_k, \mathcal{B}_k) \mid k \in \mathbb{N}\}$  is an antichain, as desired.  $\square$

**Lemma 6.3.** *For every finite group  $\Gamma$  and every  $t \geq 3$  there does not exist an infinite antichain in  $\mathcal{G}_\Gamma$  of biased graphs on  $t$  vertices.*

*Proof.* Let  $G_1, G_2, \dots$  be an infinite sequence of graphs on the vertex set  $\{1, 2, \dots, t\}$  and without loss of generality assume that every edge with ends  $i, j$  with  $i < j$  is oriented from  $i$  to  $j$ . For every  $k$  let  $\phi_k : E(G_k) \rightarrow \Gamma$  be a function. Let  $\Gamma = \{g_1, \dots, g_\ell\}$  and proceed as follows. For every  $G_k$  consider the number of edges between 1 and 2 with label  $g_1$ . This is an infinite sequence of nonnegative integers, so it has an infinite non-decreasing subsequence. Now restrict the original sequence of graphs to the corresponding subsequence. Continuing in this manner for each group element, and then repeating this process for every pair of vertices yields an infinite sequence of group labelled graphs each contained in the next.  $\square$

*Proof of Theorem 1.5:* This is an immediate consequence of the previous two lemmas.

Turning our attention to matroids, Lemma 5.2 shows that the biased graphs used in the construction from Lemma 6.2 are both frame and lift unique assuming they each have at least four edges between each pair of vertices (as we may). This immediately gives the following corollary.

**Corollary 6.4.** *For every infinite group  $\Gamma$  and every  $t \geq 3$ , there exist infinite antichains of rank  $t$  matroids in both  $\mathcal{L}_\Gamma$  and  $\mathcal{F}_\Gamma$ .*

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